

Chiral Algebras of $(0, 2)$ Models

Junya Yagi

Center for Frontier Science, Chiba University

Chiba 263–8522 Japan

yagi@cfs.chiba-u.ac.jp

Abstract

We explore two-dimensional sigma models with $(0, 2)$ supersymmetry through their chiral algebras. Perturbatively, the chiral algebras of $(0, 2)$ models have a rich infinite-dimensional structure described by the cohomology of a sheaf of chiral differential operators. Nonperturbatively, instantons can deform this structure drastically. We show that under some conditions they even annihilate the whole algebra, thereby triggering the spontaneous breaking of supersymmetry. For a certain class of Kähler manifolds, this suggests that there are no harmonic spinors on their loop spaces and gives a physical proof of the Höhn–Stolz conjecture.

1 Introduction

Supersymmetric sigma models in two dimensions have played central roles in a number of important physical and mathematical developments during the past few decades. A key concept underlying much of the developments is that of the chiral rings of sigma models with $(2, 2)$ supersymmetry [1]. These finite-dimensional cohomology rings are basic ingredients of topological sigma models [2, 3], and intimately connected to, among other things, Gromov–Witten invariants [4, 2], Floer homology [5, 6, 7, 8, 9], and mirror symmetry [10, 1, 11].

While the chiral rings of $(2, 2)$ models are clearly very interesting, we also know that some of the beautiful structures of two-dimensional supersym-

metric sigma models arise in essentially infinite-dimensional contexts. For example, the elliptic genera encode infinite series of topological invariants of the target space [12, 13]. It is then natural to ask whether there are infinite-dimensional analogs of the chiral rings. The answer is “yes.” They are the *chiral algebras* of sigma models with $(0, 2)$ supersymmetry [14, 15].

The chiral algebra of a $(0, 2)$ model is the cohomology of local operators with respect to one of the supercharges, graded by the right-moving R-charge and equipped with operator product expansions (OPEs). As a consequence of $(0, 2)$ supersymmetry, its elements vary holomorphically on the worldsheet and form a structure analogous to the chiral algebra of a conformal field theory (CFT), the operator product algebra of holomorphic fields. It is well known that one can twist $(2, 2)$ models to obtain topological field theories characterized by their chiral rings. For $(0, 2)$ models, one can perform a similar quasi-topological twisting which turns them into holomorphic field theories characterized by their chiral algebras.

Classically, the chiral algebra of a twisted $(0, 2)$ model is isomorphic, as a graded vector space, to the direct sum of the Dolbeault cohomology groups of a certain infinite series of holomorphic vector bundles over the target space. Quantum mechanically, this isomorphism gets deformed by quantum corrections. Like the chiral rings, the chiral algebra is independent of the choice of the metric on the target space, hence can be computed in the large volume limit where the theory is weakly coupled and the path integral localizes to instantons. But unlike the chiral rings, it receives perturbative corrections as well as instanton corrections, because the contributions from bosonic and fermionic fluctuations do not cancel due to the lack of left-moving supersymmetry. This complication leads to the interesting subject of perturbative chiral algebras.

At the level of perturbation theory, the physics of a sigma model is governed by the local geometry of the target space. On the other hand, we can always deform the target space metric to make it locally flat without affecting the chiral algebra. Combining these observations, we reach a surprising conclusion: the perturbative chiral algebra can be described locally by a sigma model with flat target space, therefore reconstructed by gluing locally defined *free* theories globally over the target space.

This fact was exploited by Witten [16] to show that in the absence of left-moving fermions, the perturbative chiral algebra of a twisted $(0, 2)$ model can be formulated as the cohomology of a sheaf of chiral differential operators on the target space, a notion introduced earlier in mathematics by Malikov

et al. [17]. In this picture, the moduli of the perturbative chiral algebra are encoded in the different possible ways of gluing relevant free CFTs, whereas the anomalies of the theory manifest themselves in the obstructions to doing so consistently. When left-moving fermions are present and take values in the tangent bundle of the target space, it was shown by Kapustin [18] that the perturbative chiral algebra is given by the cohomology of the chiral de Rham complex [17]. The theory of perturbative chiral algebras has been further developed along these lines by Tan [19, 20, 21, 22].

Nonperturbatively, instantons can change the picture radically [16, 23, 24, 25]. A particularly striking example is the model with no left-moving fermions whose target space is the flag manifold G/B of a complex simple Lie group G . The perturbative chiral algebra of this model is infinite-dimensional and has the structure of a $\widehat{\mathfrak{g}}$ -module of critical level [17, 26, 27]. In the presence of instantons, however, the equation $1 = 0$ holds in the cohomology and the chiral algebra *vanishes*.

One clue to the existence of such a nonperturbative phenomenon lies in a conjecture made independently by Höhn and Stolz [28] in the mid 1990s. The Höhn–Stolz conjecture asserts that the elliptic genus of a supersymmetric sigma model with no left-moving fermions vanishes if the target space M admits a Riemannian metric of positive Ricci curvature.

Stolz gave a heuristic argument for his conjecture based on the geometry of the loop space $\mathcal{L}M$, the space of smooth maps from the circle S^1 to M . It goes as follows. Let us assume that the scalar curvature of $\mathcal{L}M$ is given, at each loop $\gamma \in \mathcal{L}M$, by the integral of the Ricci curvature of M along γ . Then $\mathcal{L}M$ has positive scalar curvature if M has positive Ricci curvature. By analogy with the Lichnerowicz theorem, this would imply that $\mathcal{L}M$ has no harmonic spinors. Meanwhile, supersymmetric states of the theory may be identified with harmonic spinors on $\mathcal{L}M$. Hence the theory would have no supersymmetric states, and since the elliptic genus counts the number of bosonic supersymmetric states minus the number of fermionic ones at each energy level, it would vanish then.

If Stolz’s reasoning is correct, the positivity of the Ricci curvature implies not only the vanishing of the elliptic genus, but also the spontaneous breaking of supersymmetry. Flag manifolds have positive Ricci curvature, so supersymmetry should be broken in the models into these spaces. Supersymmetry breaking is indeed triggered whenever the chiral algebra vanishes—instantons tunnel between infinitely many perturbative supersymmetric states and lift all of them at once.

In fact, what happens for the flag manifold model is a special case of a more general phenomenon: the chiral algebra of a $(0, 2)$ model with no left-moving fermions vanishes nonperturbatively if the target space is a compact Kähler manifold with positive first Chern class and contains an embedded \mathbb{P}^1 with trivial normal bundle.

This vanishing theorem—or rather, “theorem” with quotation marks—is the main result of this paper, which we will “prove” by a physical argument. As a “corollary,” the “theorem” implies that supersymmetry is spontaneously broken. Therefore, for this particular class of Kähler manifolds, it suggests that there are no harmonic spinors on their loop spaces and gives a physical proof of the Höhn–Stolz conjecture.

The paper is organized as follows. In Section 2, we introduce the chiral algebras of $(0, 2)$ models and discuss their general properties. Section 3 is devoted to the sheaf theory of perturbative chiral algebras. Finally, in Section 4, we establish the vanishing “theorem” and explain its relation to supersymmetry breaking and the geometry of loop spaces.

2 Chiral Algebras of $(0, 2)$ Models

We now begin our study of the chiral algebras of $(0, 2)$ models. The focus of this section is on general properties of these algebras that do not depend on specific details of the target space geometry.

2.1 $(0, 2)$ Models

Two-dimensional sigma models have $(0, 2)$ supersymmetric extensions if the target space is strong Kähler with torsion (strong KT) [16]. A hermitian manifold is called strong KT if the $(1, 1)$ -form ω associated to the hermitian metric satisfies $\partial\bar{\partial}\omega = 0$. Kähler manifolds are strong KT since the Kähler form satisfies $\partial\omega = \bar{\partial}\omega = 0$. In this paper we will only consider $(0, 2)$ models with Kähler target spaces. Let us review how these models are constructed.

Let Σ be a Riemann surface and X a Kähler manifold of complex dimension d . The bosonic sigma model with worldsheet Σ and target space X is a quantum field theory of maps $\phi: \Sigma \rightarrow X$. Imposing $(0, 2)$ supersymmetry requires the introduction of two right-moving fermions, ψ_+ and $\bar{\psi}_+$. These are worldsheet spinors with values respectively in the holomorphic and

antiholomorphic tangent bundles of X :

$$\psi_+ \in \Gamma(\bar{K}_\Sigma^{1/2} \otimes \phi^* T_X), \quad \bar{\psi}_+ \in \Gamma(\bar{K}_\Sigma^{1/2} \otimes \phi^* \bar{T}_X). \quad (1)$$

Here $\bar{K}_\Sigma^{1/2}$ is a square root of the antiholomorphic canonical bundle of Σ . The simplest $(0, 2)$ model is constructed with this minimally $(0, 2)$ supersymmetric field content.

In the field space, $(0, 2)$ supersymmetry is realized as the transformation

$$\begin{aligned} \delta\phi^i &= -\epsilon_- \psi_+^i, & \delta\phi^{\bar{i}} &= \bar{\epsilon}_- \bar{\psi}_+^{\bar{i}}, \\ \delta\psi_+^i &= i\bar{\epsilon}_- \partial_{\bar{z}} \phi^i, & \delta\bar{\psi}_+^{\bar{i}} &= -i\epsilon_- \partial_z \phi^{\bar{i}}, \end{aligned} \quad (2)$$

where ϵ_- and $\bar{\epsilon}_-$ are sections of $\bar{K}_\Sigma^{-1/2}$. We define the right-moving supercharges Q_+ and \bar{Q}_+ so that $-i\epsilon_- Q_+ + i\bar{\epsilon}_- \bar{Q}_+$ acts by this transformation. The supercharges then satisfy

$$\begin{aligned} \{Q_+, Q_+\} &= \{\bar{Q}_+, \bar{Q}_+\} = 0, \\ \{Q_+, \bar{Q}_+\} &= -i\partial_{\bar{z}} = \frac{1}{2}(H - P), \end{aligned} \quad (3)$$

and generate the $(0, 2)$ supersymmetry algebra together with the generators H , P of translations, M of rotations, and F_R of the right-moving $U(1)$ R-symmetry. Under the last symmetry, ψ_+ has charge -1 and $\bar{\psi}_+$ has charge $+1$; thus Q_+ has charge -1 and \bar{Q}_+ has charge $+1$.

To construct a $(0, 2)$ supersymmetric action, we choose a Kähler metric g on X and make the operator $g_{i\bar{j}}\psi_+^i \partial_z \phi^{\bar{j}}$ of R-charge -1 . Then the action

$$S = \int_\Sigma d^2z \{ \bar{Q}_+, g_{i\bar{j}}\psi_+^i \partial_z \phi^{\bar{j}} \} \quad (4)$$

is invariant under the R-symmetry and, by virtue of the relation $\bar{Q}_+^2 = 0$, under the symmetry generated by $i\bar{\epsilon}_- \bar{Q}_+$ provided that $\bar{\epsilon}_-$ is antiholomorphic and so commutes with the ∂_z inside. This action is also invariant under $-i\epsilon_- \bar{Q}_+$ for antiholomorphic ϵ_- , as becomes clear if we rewrite it as

$$S = \int_\Sigma d^2z \{ Q_+, g_{i\bar{j}} \partial_z \phi^i \bar{\psi}_+^{\bar{j}} \}. \quad (5)$$

Expanding the anticommutators and using the Kähler condition, one can check that the two expressions (4) and (5) both coincide with

$$S = \int_\Sigma d^2z (g_{i\bar{j}} \partial_{\bar{z}} \phi^i \partial_z \phi^{\bar{j}} + i g_{i\bar{j}} \psi_+^i D_z \bar{\psi}_+^{\bar{j}}). \quad (6)$$

Here the covariant derivative D_z is the ∂ operator coupled to the pullback of the Levi-Civita connection Γ on X . Explicitly, $D_z \bar{\psi}_+^i = \partial_z \bar{\psi}_+^i + \partial_z \phi^{\bar{j}} \Gamma_{\bar{j}k}^i \bar{\psi}_+^k$.

We can add a topological invariant to the action. For a closed two-form B on X , the functional

$$S_B = \int_{\Sigma} \phi^* B \quad (7)$$

depends only on the cohomology class of B and the homotopy class of ϕ . As such, it is invariant under any continuous transformations, especially the supersymmetry transformation and the R-symmetry. The topological invariant S_B vanishes at the level of perturbation theory where one deals with homotopically trivial maps, but affects the dynamics nonperturbatively.

We have obtained a $(0, 2)$ supersymmetric action. To complete the construction, we need to make sure that a sensible quantum theory based on this action exists. It turns out that X must satisfy two topological conditions for that. First, X must be spin, or equivalently, its first Chern class must be even:

$$c_1(X) \equiv 0 \pmod{2}. \quad (8)$$

As we will see, this condition ensures that the fermion parity $(-1)^{F_R}$ is well defined. Second, the second Chern character $\text{ch}_2(X) = c_1(X)^2/2 - c_2(X)$, which is also a half of the first Pontryagin class $p_1(X)$, must be zero:

$$\frac{1}{2} p_1(X) = 0. \quad (9)$$

This is the condition for the absence of sigma model anomaly [29, 30], the obstruction to finding a well-defined path integral measure. From the viewpoint of the loop space $\mathcal{L}X$, the first condition means that $\mathcal{L}X$ is orientable [31, 32], while the second condition (given the first) is interpreted as the condition for $\mathcal{L}X$ to admit spinors [33].

There is also a geometric condition. The renormalization group generates a flow of the target space metric. At the one-loop level, the metric $g(\mu)$ renormalized at an energy scale μ obeys the equation

$$\mu \frac{d}{d\mu} g_{i\bar{j}}(\mu) = \frac{1}{2\pi} R_{i\bar{j}}(g(\mu)), \quad (10)$$

where $R_{i\bar{j}}(g)$ is the Ricci curvature of g [34, 35, 36, 37]. From this equation, we see that $g(\mu)$ gets larger as μ gets larger if the Ricci curvature is positive. Since the theory is weakly coupled when the target space has large volume,

we have asymptotic freedom in this case. In contrast, if the Ricci curvature is negative, the theory is strongly coupled for large μ and does not have a well-defined ultraviolet limit. Hence, for the microscopic theory to exist, the Ricci curvature should be semipositive.

When the above conditions are satisfied, the action (6) plus (7) defines the simplest version of $(0, 2)$ models. The supercharges are given by

$$Q_+ = \oint d\bar{z} g_{i\bar{j}} \psi_+^i \partial_{\bar{z}} \phi^{\bar{j}}, \quad \bar{Q}_+ = \oint d\bar{z} g_{i\bar{j}} \partial_{\bar{z}} \phi^i \bar{\psi}_+^{\bar{j}} \quad (11)$$

and satisfy the reality condition $Q_+^\dagger = \bar{Q}_+$.

If one has a holomorphic vector bundle E over X , one can extend the model by adding left-moving fermions with values in E . This extended model is called the heterotic model. Since the left-movers contribute to sigma model anomalies in the opposite way as the right-movers do, their presence changes the anomaly cancellation condition to

$$\frac{1}{2}p_1(X) = \frac{1}{2}p_1(E). \quad (12)$$

This is trivially satisfied if $E = T_X$, in which case the model actually has $(2, 2)$ supersymmetry. It is also possible to add superpotentials [38]. For brevity, we will not consider these extensions in this paper. We refer to Tan [19] for the perturbative aspects of the heterotic model.

2.2 Chiral Algebras

Having formulated $(0, 2)$ models, let us introduce the notion of their chiral algebras. Among the two supercharges we will use \bar{Q}_+ exclusively, so simply write $Q = \bar{Q}_+$. Then $Q^\dagger = Q_+$.

Consider the action of Q on local operators \mathcal{O} given by supercommutator $[Q, \mathcal{O}]$. The Q -action increases the R-charge of \mathcal{O} by one, and squares to zero. Given a $(0, 2)$ model, therefore, we can define the Q -cohomology of local operators graded by the R-charge.

Since $\partial_{\bar{z}} \propto H - P$ is Q -exact by the $(0, 2)$ supersymmetry algebra, it acts trivially in the Q -cohomology: if \mathcal{O} is Q -closed, then $\partial_{\bar{z}} \mathcal{O}$ is Q -exact. Thus Q -cohomology classes vary holomorphically on Σ . Moreover, two classes can be multiplied by $[\mathcal{O}] \cdot [\mathcal{O}'] = [\mathcal{O}\mathcal{O}']$. From these facts, it follows that the

Q -cohomology of local operators inherits a holomorphic OPE structure from the underlying theory:

$$[\mathcal{O}(z)] \cdot [\mathcal{O}'(w)] \sim \sum_k c_k(z-w) [\mathcal{O}_k(w)]. \quad (13)$$

The coefficient functions $c_k(z-w)$ are holomorphic away from the diagonal $z=w$ where there can be poles.

The holomorphic Q -cohomology of local operators, equipped with this natural OPE structure, is the chiral algebra of the $(0,2)$ model. We will denote the chiral algebra by \mathcal{A} , and its R -charge q subspace by \mathcal{A}^q . As is clear from the construction, the chiral algebra of a $(0,2)$ model has the structure of a chiral algebra in the sense of CFT, except that the grading by conformal weight is missing. We will see later that the chiral algebra carries a similar (but possibly reduced to \mathbb{Z}_n) grading after the theory is twisted.

The chiral algebra forms a closed sector of a $(0,2)$ model in the following sense. Consider the n -point function of Q -closed local operators:

$$\langle \mathcal{O}_1(z_1, \bar{z}_1) \cdots \mathcal{O}_n(z_n, \bar{z}_n) \rangle. \quad (14)$$

If one of the operators is Q -exact, $\mathcal{O}_i = [Q, \mathcal{O}'_i]$, then the n -point function becomes $\pm \langle [Q, \mathcal{O}_1 \cdots \mathcal{O}'_i \cdots \mathcal{O}_n] \rangle$. Computed with a Q -invariant action and path integral measure, this is the integral of a “ Q -exact form” on the field space and vanishes. The n -point function (14) thus depends only on the Q -cohomology classes $[\mathcal{O}_i]$. In particular, it is a holomorphic (more precisely, meromorphic) function of the insertion points.

So far, we have considered the chiral algebra of a fixed $(0,2)$ model, defined by a fixed Q -invariant action. Of course, different choices of the action lead to different chiral algebras in general. Imagine deforming the theory by perturbing the action:

$$S \rightarrow S + \delta S. \quad (15)$$

For this deformation to preserve the Q -invariance, δS must be Q -closed. If, however, δS is Q -exact, the chiral algebra remains unchanged. To see this, express the matrix elements of Q as path integrals on a cylinder of infinitesimal length, with a contour integral of the supercurrent sandwiched between various boundary conditions. One can show that for a Q -exact perturbation, the matrix elements of Q in the deformed theory are equal to

those of $Q + [Q, \delta A]$ in the original theory for some operator δA . But this latter operator is related to Q by the conjugation

$$Q \rightarrow e^{-\delta A} Q e^{\delta A}, \quad (16)$$

hence defines an isomorphic chiral algebra. The n -point function (14) is also unchanged because the perturbation just introduces Q -exact insertions.

Looking back at the action (4) of our model, we see that deformations of the target space metric just give Q -exact perturbations. Therefore, the chiral algebra is independent of the Kähler structure of the target space. It does depend on the complex structure, however. For this enters the very definition of the supersymmetry transformation.

2.3 Instantons

An important property of the chiral algebra is that it receives contributions only from instantons and small fluctuations around them. In the case of our model, instantons obey

$$\{Q, \psi_+^i\} = \partial_{\bar{z}} \phi^i = 0, \quad (17)$$

so they are holomorphic maps from Σ to X . It is this localization principle that makes the chiral algebra effectively computable.

To prove the localization, we rescale the target space metric so that it becomes very large. In this large volume limit, the path integral localizes to the zeros of the bosonic action

$$\int_{\Sigma} d^2 z \, g_{i\bar{j}} \partial_{\bar{z}} \phi^i \partial_z \phi^{\bar{j}}, \quad (18)$$

namely holomorphic maps, or instantons, as promised. Incidentally, there is another situation in which the same localization arises. That is when the path integral computes the correlation function of Q -closed operators [3]. This situation is not really relevant for us, though. In order to determine the chiral algebra, we have to ask, in the first place, whether a given local operator is Q -closed or not.

The space \mathcal{M} of holomorphic maps from Σ to X is called the instanton moduli space. It decomposes into disconnected components labeled by the homology class β that the image of Σ represents in X :

$$\mathcal{M} = \bigoplus_{\beta \in H_2(X, \mathbb{Z})} \mathcal{M}_{\beta}. \quad (19)$$

Correspondingly, the path integral splits into distinct sectors each of which integrates over the neighborhood of a single connected component of \mathcal{M} . In sigma model perturbation theory, one expands in the inverse volume of the target space in the zero-instanton sector. Instantons of $\beta = 0$ are constant maps, whose moduli space $\mathcal{M}_0 \cong X$.

When $c_1(X) \neq 0$, instantons induce two important nonperturbative effects. One is the violation of the R-charge conservation, which breaks the R-symmetry down to a discrete subgroup. The other is the appearance of powers of a dynamical scale Λ generated via dimensional transmutation, which allows objects of different scaling dimensions to show up as quantum corrections.

The anomaly in the R-symmetry is due to a nontrivial transformation of the fermionic path integral measure under this symmetry. The complex conjugate of $\bar{\psi}_+$ is a section of $K_\Sigma^{1/2} \otimes \phi^* T_X$, while ψ_+ can be identified with a $(0, 1)$ -form with values in the same bundle. Thus, we can define $\bar{D} = D_{\bar{z}} d\bar{z}$ and expand these fields in the eigenmodes of $\bar{D}^* \bar{D}$ and $\bar{D} \bar{D}^*$:

$$\psi_+ = \sum_s b_0^s v_{0,s} + \sum_n b^n v_n, \quad \bar{\psi}_+ = \sum_r c_0^r \bar{u}_{0,r} + \sum_n c^n \bar{u}_n, \quad (20)$$

Here $\bar{u}_{0,r}$, $v_{0,s}$ are zero modes, \bar{u}_n , v_n nonzero modes, and b_0^s , c_0^r , b^n , c^n anticommuting coefficients. The fermionic path integral measure is the formal product

$$\prod_{r,s,n} db_0^s dc_0^r db^n dc^n. \quad (21)$$

The nonzero mode part of the measure is neutral under the R-symmetry because of the pairing of nonzero modes. The zero mode part has R-charge equal to the number of ψ_+ zero modes minus the number of $\bar{\psi}_+$ zero modes, that is, minus the index of the \bar{D} operator. For a compact Riemann surface Σ , the index is given by

$$\int_\Sigma \phi^* c_1(X) \quad (22)$$

which (recalling $c_1(X) \equiv 0 \pmod{2}$) is equal to $2k$ for some integer k . The R-charge is violated by this amount in an instanton background. We see that the R-symmetry is broken to a \mathbb{Z}_{2n} subgroup in the presence of instantons, where n is the greatest common divisor of the integers k .

Instantons produce powers of Λ when $c_1(X) \neq 0$ because the B field is renormalized as

$$[B(\mu)] = [B_0] + \ln \frac{\mu}{\Lambda} c_1(X), \quad (23)$$

with $[B_0]$ a fixed class in $H^2(X, \mathbb{C})$. We will derive this formula at the end of this section. Accordingly, instanton corrections violating R-charge by $2k$ units are weighted by the topological factor

$$e^{-S_B} = \left(\frac{\Lambda}{\mu}\right)^{2k} \exp\left(\int_{\Sigma} \phi^* B_0\right). \quad (24)$$

An extra factor of μ^{2k} should come from somewhere else to cancel the dependence on μ . Altogether, these corrections are proportional to Λ^{2k} and can relate objects of scaling dimensions differing by $2k$.

For instanton corrections to be under control, we should choose B_0 such that the factor (24) is exponentially suppressed for nonconstant holomorphic maps. The Kähler form is a good choice, for example.

2.4 Twisting

The action of our model is invariant under supersymmetries whose transformation parameters are antiholomorphic sections of the bundle $\overline{K}_{\Sigma}^{-1/2}$. If Σ is topologically nontrivial, this bundle may not admit any global antiholomorphic sections. In that case the supersymmetries exist at best locally on Σ , hence so does the chiral algebra. However, what we need to define the chiral algebra is really one of the two supersymmetries, not both. So it would be nice if we can somehow globalize one in return for giving up the other. This is achieved by twisting the theory.

Let us modify the spins of the fermionic fields so that ψ_+ becomes a $(0, 1)$ -form on Σ , while $\bar{\psi}_+$ becomes a zero-form. To make this point clear, we rename them as

$$-\psi_+^i \rightarrow \rho_{\bar{z}}^i, \quad -i\bar{\psi}_+^{\bar{i}} \rightarrow \alpha^{\bar{i}}. \quad (25)$$

The action of Q is then given by

$$\begin{aligned} [Q, \phi^i] &= 0, & [Q, \phi^{\bar{i}}] &= \alpha^{\bar{i}}, \\ \{Q, \rho_{\bar{z}}^i\} &= -\partial_{\bar{z}} \phi^i, & \{Q, \alpha^{\bar{i}}\} &= 0. \end{aligned} \quad (26)$$

Thus Q becomes a scalar and generates a global supersymmetry. On the other hand, Q_+ becomes a $(0, 1)$ -form and in general does not exist globally.

In this way, we obtain a theory with global supersymmetry on any Riemann surface Σ , described by the action

$$S = \int_{\Sigma} d^2z \{Q, g_{i\bar{j}} \rho_{\bar{z}}^i \partial_z \phi^{\bar{j}}\} + \int_{\Sigma} \phi^* B. \quad (27)$$

Discarding half the original supersymmetries also allows us to consider complex target spaces which may or may not be Kähler.

Besides globalizing the supersymmetry, the twisting does one more important thing: it makes the components $T_{z\bar{z}}$ and $T_{\bar{z}z}$ of the energy-momentum tensor Q -exact. This is because the global supersymmetry commutes with infinitesimal diffeomorphisms $\delta z = v^z$, $\delta \bar{z} = v^{\bar{z}}$ up to a quantity involving $\partial_{\bar{z}} v^z$, but not $\partial_{\bar{z}} v^{\bar{z}}$, $\partial_z v^z$, or $\partial_z v^{\bar{z}}$. Hence, one can take the variation of the action inside the Q -commutator when computing these components. Quantum mechanically the action gets renormalized, but the renormalization can be done by Q -exact local counterterms and the conclusion is unchanged.¹

The generators $T_{z\bar{z}}$, $T_{\bar{z}z}$ being Q -exact, antiholomorphic reparametrizations act trivially in the Q -cohomology. Therefore, after the twisting the chiral algebra has no antiholomorphic degrees of freedom—it defines a holomorphic field theory.

A local operator \mathcal{O} is said to have dimension (n, m) if inserted at the origin it transforms as

$$\mathcal{O}(0) \rightarrow \lambda^{-n} \bar{\lambda}^{-m} \mathcal{O}(0) \quad (28)$$

under the rescaling $z \rightarrow \lambda z$, $\bar{z} \rightarrow \bar{\lambda} \bar{z}$. An immediate consequence of the decoupling of antiholomorphic degrees of freedom is that the chiral algebra is supported by local operators with $m = 0$, because those with $m \neq 0$ transform nontrivially under antiholomorphic rescalings. The holomorphic dimension, n , is then equal to the spin $n - m$, so it is an integer and protected from quantum corrections (assuming that they are small). Thus, the chiral algebra of the twisted model is graded by the dimension as well as the R-charge.

If $c_1(X) \neq 0$, the grading by the R-charge is violated by instantons as we discussed already. The same is true of the grading by dimension. Instanton corrections are accompanied with powers of Λ^2 . When multiplying a local

¹This is not a precise statement. When Σ is curved, one needs to introduce a worldsheet metric to impose a meaningful cutoff length. This produces an anomalous term in $T_{z\bar{z}}$ proportional to the Ricci curvature of Σ [37]. Such a c-number anomaly does not affect the structure of the chiral algebra.

operator in the untwisted model, Λ^2 is best thought of as a section of $K_\Sigma \otimes \bar{K}_\Sigma$ having dimension $(1, 1)$. In the twisted model, it is more natural to think of it as a section of K_Σ with dimension $(1, 0)$, absorbing the change in the dimension of the fermionic path integral measure. (This can be deduced by considering the case where $K_\Sigma^{1/2}$ is trivial and so the twisted and untwisted models are isomorphic.) Instanton corrections violating R-charge by $2k$ units are proportional to Λ^{2k} , so violate dimension by k . The grading by dimension is therefore reduced to \mathbb{Z}_n nonperturbatively when that by the R-charge is reduced to \mathbb{Z}_{2n} .

With the grading by dimension at hand, we can now write the OPE of the chiral algebra in the form

$$[\mathcal{O}_i(z)] \cdot [\mathcal{O}_j(w)] \sim \sum_{n=0}^{\infty} \sum_k \frac{\Lambda^{2n} c_{ijk}^{(n)} [\mathcal{O}_k(w)]}{(z-w)^{n_i+n_j-n_k-n}}, \quad (29)$$

where $[\mathcal{O}_i]$ have dimension n_i and $c_{ijk}^{(n)}$ are constants. The Q -cohomology group of dimension n vanishes for $n < 0$ since there are no local operators of negative dimension classically.

The Q -cohomology classes of dimension zero are special: they have regular OPEs, as one can see by setting $n_i = n_j = 0$ above and noting $n_k \geq 0$. Hence they form an ordinary ring, much like the chiral rings of $(2, 2)$ models. This ring is called the chiral ring of the $(0, 2)$ model [16, 39].

We have seen advantages of the twisting. It has one drawback, however. Twisting the fermions amounts to tensoring with $\bar{K}_\Sigma^{-1/2}$ to which the ∂ operator couples. This changes the anomaly cancellation condition: Σ and X must now satisfy

$$\frac{1}{2} p_1(X) = \frac{1}{2} c_1(\Sigma) c_1(X) = 0. \quad (30)$$

Here $c_1(\Sigma)$ and $c_1(X)$ are pulled back to $\Sigma \times X$. Thus, the twisting introduces an additional anomaly which affects the choice of Σ . If $c_1(X) \neq 0$, we must choose Σ with $c_1(\Sigma) = 0$, that is to say, K_Σ must be trivial. This was actually implicit in our discussion when we interpreted Λ^2 as a nowhere vanishing section of K_Σ .

2.5 Classical Chiral Algebra and Quantum Deformations

To determine the chiral algebra of a given model, what one can usually do is to first identify its elements in the classical limit, then include quantum corrections order by order. What guarantees the validity of this procedure is the following principle: small quantum corrections can only annihilate, and never create, Q -cohomology classes.

This statement is justified as follows. Local operators that are not Q -closed at a given order cannot become Q -closed by higher order corrections. So the kernel of Q can never become larger. At the same time, the image of Q can never become smaller, because adding quantum corrections order by order defines an injection from the image of Q at a given order to the image of Q at the next order. Then the cohomology, which is the kernel modulo the image, can only become smaller.

In this respect, the chiral algebra is similar to the space of supersymmetric ground states (of supersymmetric quantum mechanics, say). There, small quantum corrections can lift supersymmetric ground states by giving a small positive energy, but not create ones because they cannot push positive energy states down to zero energy as long as there is a gap in the beginning. Likewise, small quantum corrections can “lift” Q -cohomology classes by making them Q -exact or no longer Q -closed, but not create ones because of a “gap” to the Q -closedness or non- Q -exactness.

The analogy goes further. Supersymmetry yields a one-to-one correspondence between the bosonic positive energy states and the fermionic ones, so supersymmetric ground states are always lifted in boson-fermion pairs. The same principle applies here: small quantum corrections always annihilate Q -cohomology classes in boson-fermion pairs. More precisely, one can show that when a Q -cohomology class $[\mathcal{O}]$ is annihilated by quantum corrections at some order, there is another class $[\mathcal{O}']$ that is annihilated together at the same order either via the relation $[Q, \mathcal{O}] \propto \mathcal{O}'$ or $\mathcal{O} \propto [Q, \mathcal{O}']$.²

²To prove this, let ϵ be a small parameter that controls the strength of quantum effects, and \mathcal{O} a local operator that is nontrivial in the Q -cohomology to order ϵ^{k-1} . First, suppose that \mathcal{O} is no longer Q -closed at order ϵ^k ; thus $[Q, \mathcal{O}] = \epsilon^k \mathcal{O}'$ for some \mathcal{O}' , and no corrections to \mathcal{O} can make it Q -closed again. Then \mathcal{O}' is Q -closed, but cannot be Q -exact at order ϵ^l for any $l < k$. For if there exists \mathcal{O}'' such that $\epsilon^l \mathcal{O}' = [Q, \mathcal{O}''] + O(\epsilon^{l+1})$, then $[Q, \mathcal{O} - \epsilon^{k-l} \mathcal{O}''] = O(\epsilon^{k+1})$ and $\mathcal{O} - \epsilon^{k-l} \mathcal{O}''$ is Q -closed to order ϵ^k . Next, suppose that \mathcal{O} becomes Q -exact at order ϵ^k ; thus $\epsilon^k \mathcal{O} = [Q, \mathcal{O}'] + O(\epsilon^{k+1})$ for some \mathcal{O}' . Then \mathcal{O}' is

Keeping the above principles in mind, let us study the chiral algebra of the twisted model in a little more detail.

The chiral algebra is supported by local operators whose antiholomorphic dimension $m = 0$. Since ρ and \bar{z} -derivatives of any fields have $m > 0$, these do not enter the chiral algebra. Furthermore, we can replace $\partial_z \alpha$ by other fields using the equation of motion $D_z \alpha^{\bar{i}} = 0$. Thus, relevant local operators are linear combinations of operators of the form

$$\mathcal{O}(\phi, \bar{\phi})_{j \dots k \dots \bar{l} \dots \bar{m} \dots \bar{i}_1 \dots \bar{i}_q} \partial_z \phi^j \dots \partial_z^2 \phi^k \dots \partial_z \phi^{\bar{l}} \dots \partial_z^2 \phi^{\bar{m}} \dots \alpha^{\bar{i}_1} \dots \alpha^{\bar{i}_q}. \quad (31)$$

Identifying $\alpha^{\bar{i}}$ with the differential $d\phi^{\bar{i}}$, we can regard such operators with R-charge q and dimension n as $(0, q)$ -forms with values in a certain holomorphic vector bundle $V_{X,n}$ over X .

For example, an $n = 0$ operator $\mathcal{O}_{\bar{i}_1 \dots \bar{i}_q} \alpha^{\bar{i}_1} \dots \alpha^{\bar{i}_q}$ is a $(0, q)$ -form on X , hence $V_{X,0} = 1$, the trivial bundle of rank 1. At $n = 1$, there are two types of operators, $\mathcal{O}_{j \bar{i}_1 \dots \bar{i}_q} \partial_z \phi^j \alpha^{\bar{i}_1} \dots \alpha^{\bar{i}_q}$ and $\mathcal{O}_{\bar{i}_1 \dots \bar{i}_q}^j g_{j \bar{k}} \partial_z \phi^{\bar{k}} \alpha^{\bar{i}_1} \dots \alpha^{\bar{i}_q}$. These are $(0, q)$ -forms with values respectively in T_X^* and T_X , thus $V_{X,1} = T_X \oplus T_X^*$. At $n = 2$, we have five types, giving $V_{X,2} = T_X \oplus T_X^* \oplus S^2 T_X \oplus (T_X \otimes T_X^*) \oplus S^2 T_X^*$. Here S^k is the symmetric k th power. In general, $V_{X,n}$ is given by the series

$$\sum_{n=0}^{\infty} q^n V_{X,n} = \bigotimes_{k=0}^{\infty} S_{q^k}(T_X) \bigotimes_{l=0}^{\infty} S_{q^l}(T_X^*), \quad (32)$$

where $S_t(V) = 1 + tS(V) + t^2 S^2(V) + \dots$. We will write

$$V_X = \bigoplus_{n=0}^{\infty} V_{X,n}. \quad (33)$$

At the classical level, we can easily find the action of Q on these operators. Take an $n = 1$ example, $\mathcal{O}_{j \bar{i}_1 \dots \bar{i}_q} \partial_z \phi^j \alpha^{\bar{i}_1} \dots \alpha^{\bar{i}_q}$. Acting on it with Q gives $\alpha^{\bar{i}} \partial_{\bar{i}} \mathcal{O}_{j \bar{i}_1 \dots \bar{i}_q} \partial_z \phi^j \alpha^{\bar{i}_1} \dots \alpha^{\bar{i}_q}$. On an $n = 1$ operator of the other type, $\mathcal{O}_{\bar{i}_1 \dots \bar{i}_q}^j g_{j \bar{k}} \partial_z \phi^{\bar{k}} \alpha^{\bar{i}_1} \dots \alpha^{\bar{i}_q}$, acting with Q gives $\alpha^{\bar{i}} \partial_{\bar{i}} \mathcal{O}_{\bar{i}_1 \dots \bar{i}_q}^j g_{j \bar{k}} \partial_z \phi^{\bar{k}} \alpha^{\bar{i}_1} \dots \alpha^{\bar{i}_q}$ plus $\mathcal{O}_{\bar{i}_1 \dots \bar{i}_q}^j g_{j \bar{k}} D_z \alpha^{\bar{k}} \alpha^{\bar{i}_1} \dots \alpha^{\bar{i}_q}$, but the latter vanishes by the equation of motion.

Q -closed to order ϵ^{k-1} , but cannot be Q -exact to the same order. For if there exist \mathcal{O}'' and \mathcal{O}''' such that $\epsilon^l \mathcal{O}' = [Q, \mathcal{O}''] + \epsilon^{l+1} \mathcal{O}'''$ for some $l < k$, then $\epsilon^{k-1} \mathcal{O} = [Q, \mathcal{O}'''] + \mathcal{O}(\epsilon^k)$ and \mathcal{O} is already Q -exact at order ϵ^{k-1} .

From these examples, we see that Q acts as the $\bar{\partial}$ operator.³ Classically, the q th Q -cohomology group of dimension n is therefore isomorphic to the q th Dolbeault cohomology group $H_{\bar{\partial}}^q(X, V_{X,n})$, and

$$\mathcal{A} \cong \bigoplus_{q=0}^d \bigoplus_{n=0}^{\infty} H_{\bar{\partial}}^q(X, V_{X,n}) \quad (34)$$

as graded vector spaces. It is clear from this formula that the chiral algebra is generally infinite-dimensional.

Quantum mechanically, the action of Q is deformed by quantum corrections. Because of this deformation, some of the classical Q -cohomology classes can disappear.

A notable example is the disappearance of the energy-momentum tensor. Classically, our model is conformally invariant and T_{zz} is Q -closed on-shell. We expect that it is no longer Q -closed perturbatively if the Ricci curvature is nonzero, since conformal invariance is broken at one loop in that case. To determine $[Q, T_{zz}]$, we note that Q commutes with $\partial_z \propto H + P$. Thus

$$\left[Q, \oint dz T_{zz} - \oint d\bar{z} T_{\bar{z}\bar{z}} \right] = \oint dz [Q, T_{zz}] = 0, \quad (35)$$

where we have used the fact that $T_{z\bar{z}}$ is Q -exact. This suggests

$$[Q, T_{zz}] = \partial_z \theta \quad (36)$$

for some θ . Presumably, θ is a Q -closed local operator of R-charge one and dimension one, constructed from the target space metric. To leading order, $Q = \bar{\partial}$ and for a generic choice of the metric there is only one possibility:

$$\theta \propto R_{i\bar{j}} \partial_z \phi^i \alpha^{\bar{j}}. \quad (37)$$

So, as expected, T_{zz} would cease to be Q -closed unless the Ricci curvature vanishes. In Section 3.3, we will compute $[Q, T_{zz}]$ explicitly in perturbation theory and find that it is indeed proportional to $\partial_z(R_{i\bar{j}} \partial_z \phi^i \alpha^{\bar{j}})$ up to higher order corrections.

³General local operators of dimension greater than one are constructed using covariant derivatives. On such operators, Q acts by $\bar{\partial}$ plus terms involving the curvature of the target space. Classically, these additional terms which vanish for a flat metric do not change the Q -cohomology. This point will become clear when we develop a sheaf theory approach to the perturbative chiral algebra in Section 3.

If $c_1(X) = 0$, then θ is Q -exact and we can add corrections to T_{zz} that make it Q -closed again. But if $c_1(X) \neq 0$, there is no way to do this, so $[T_{zz}]$ is annihilated together with $[\partial_z \theta]$ by perturbative corrections. The chiral algebra is a little exotic in this case: it is analogous to the chiral algebras of CFTs, but lacks the energy-momentum tensor and hence invariance under holomorphic reparametrizations.

Except for the possible lack of the energy-momentum tensor, perturbatively the structure of the chiral algebra is not very different from its classical description. The basic fact about perturbative corrections is that they are local on the target space, because one only considers fluctuations around constant maps in perturbation theory. Moreover, the gradings by the R-charge and dimension are not violated perturbatively. Thanks to these properties, at the perturbative level the action of Q still defines differential complexes

$$\cdots \longrightarrow V_{X,n} \otimes \wedge^q \overline{T}_X^* \xrightarrow{Q} V_{X,n} \otimes \wedge^{q+1} \overline{T}_X^* \longrightarrow \cdots, \quad (38)$$

and the chiral algebra is given by the direct sum of their cohomology groups. How to compute these cohomology groups using a sheaf of free CFTs on X is the subject of the next section.

Beyond perturbation theory, the physics is no longer local on the target space. Rather, it is local on the instanton moduli space \mathcal{M} since quantum fluctuations localize to instantons. So presumably the chiral algebra can be formulated nonperturbatively as a cohomology theory on \mathcal{M} , but exactly how this should be done is not clear. At any rate, in principle one can always compute the instanton corrections to the action of Q by path integral and determine the exact chiral algebra. Instanton effects often lead to surprising results. Later we will see examples where the whole chiral algebra is annihilated by instanton corrections.

Let us tie up the loose ends from Section 2.3 by explaining why the B field should be renormalized as asserted there. Consider the case where the twisted and untwisted models are isomorphic. Suppose that instantons annihilate perturbative Q -cohomology classes $[\mathcal{O}]$ and $[\mathcal{O}']$ through a relation of the form $[Q, \mathcal{O}] \sim \mathcal{O}'$. If this relation violates R-charge by $2k$ units, the left-hand side contains $2k$ more α fields than the right-hand side does, while the both sides have antiholomorphic dimension equal to zero. Viewed in the untwisted model, this means that these instantons relate operators whose antiholomorphic dimensions differ by k . Since the dynamical scale Λ is the only dimensional parameter available and has antiholomorphic dimension

$1/2$, to match the scaling dimensions a factor of Λ^{2k} must appear in the right-hand side. For that, the B field must obey the renormalization group equation (23).

3 Sheaf Theory of Perturbative Chiral Algebras

As we have explained in the last section, the chiral algebra can be understood as a quantum deformation of the Dolbeault cohomology of an infinite-dimensional holomorphic vector bundle over the target space. At the perturbative level, there is an alternative formulation of the chiral algebra which involves a sheaf of free CFTs. The goal of this section is to develop the sheaf theory of perturbative chiral algebras.

3.1 Perturbative Chiral Algebra from Free CFTs

The chiral algebra of the twisted model is classically the Dolbeault cohomology of the holomorphic vector bundle V_X . By the Čech–Dolbeault isomorphism, this cohomology is isomorphic to the cohomology of the sheaf of holomorphic sections of V_X . Perturbatively Q gets corrected, but still acts as a differential operator on the target space due to the locality of sigma model perturbation theory. In such a situation, there is an analog of the Čech–Dolbeault isomorphism: the perturbative Q -cohomology is isomorphic to the cohomology of the sheaf of perturbatively Q -closed sections of V_X . The proof is completely parallel to the classical case.⁴

This “Čech– Q isomorphism” may seem to have little practical value. To compute the sheaf cohomology, first of all one needs to know the general form of perturbatively Q -closed local sections of V_X . But this requires understanding beforehand how perturbative corrections deform the classical expression $Q = \bar{\partial}$ precisely, which is generally very hard if not impossible. However, we can circumvent this difficulty if we adopt a different approach.

⁴The key ingredients are the Q -Poincaré lemma and the existence of partitions of unity on the sheaf of $(0, q)$ -forms with values in V_X . The former follows from the $\bar{\partial}$ -Poincaré lemma since small quantum corrections cannot create Q -cohomology classes. As for the latter, “quantum” partitions of unity can always be constructed by renormalizing “classical” partitions of unity.

Let us recast the Čech– Q isomorphism in a slightly more abstract form. Choose a good cover $\{U_\alpha\}$ on X ; thus all nonempty finite intersections of U_α are diffeomorphic to \mathbb{C}^d . On each U_α , the space of perturbatively Q -closed sections of V_X is isomorphic to the chiral algebra of the twisted model into U_α . For, the zeroth Q -cohomology group is isomorphic to this space, while the higher Q -cohomology groups vanish as U_α is topologically trivial. Since Čech cohomology is defined by taking the direct limit as the open cover becomes finer and finer, and every open cover has a refinement by a good cover, it follows that the perturbative chiral algebra can be computed via the cohomology of the sheaf of chiral algebras $\hat{\mathcal{A}}$ on X :

$$\mathcal{A}^q \cong H^q(X, \hat{\mathcal{A}}). \quad (39)$$

Rewriting the Čech– Q isomorphism this way makes it clear that the perturbative chiral algebra can be formulated without reference to any globally defined metric on the target space.

When one computes the cohomology of $\hat{\mathcal{A}}$, say using a good cover $\{U_\alpha\}$, one can endow U_α with any metric. In practice, we will put $g_{i\bar{j}} = \delta_{i\bar{j}}$ on all the U_α . The point is that the twisted models with the flat target spaces U_α are free theories—their chiral algebras receive no quantum corrections.

In the free twisted model $Q = \bar{\partial}$ exactly, so sections of $\hat{\mathcal{A}}(U_\alpha)$ are represented by local operators of the form $\mathcal{O}(\phi, \partial_z \phi, \dots, \partial_z \bar{\phi}, \partial_z^2 \bar{\phi}, \dots)$. Introducing bosonic fields β_i of dimension one and γ^i of dimension zero by

$$\beta_i = 2\pi \delta_{i\bar{j}} \partial_z \bar{\phi}^{\bar{j}}, \quad \gamma^i = \phi^i, \quad (40)$$

we can conveniently write them as $\mathcal{O}(\gamma, \partial_z \gamma, \dots, \beta, \partial_z \beta, \dots)$. These are observables of the free $\beta\gamma$ system, a free CFT with action

$$S = \frac{1}{2\pi} \int_{\Sigma} d^2 z \beta_i \partial_{\bar{z}} \gamma^i \quad (41)$$

whose OPEs are

$$\beta_i(z) \gamma^j(w) \sim -\frac{\delta_i^j}{z-w}, \quad \beta_i(z) \beta_j(w) \sim 0, \quad \gamma^i(z) \gamma^j(w) \sim 0. \quad (42)$$

Hence, $\hat{\mathcal{A}}$ may be considered as a sheaf of free $\beta\gamma$ systems. We conclude that the perturbative chiral algebra can be reconstructed by gluing free $\beta\gamma$ systems over X and computing the Čech cohomology.

This construction is exact to all orders in perturbation theory since the spaces $\widehat{\mathcal{A}}(U_\alpha)$ are free of quantum corrections. Where did the perturbative corrections go? They are now encoded in the transition functions $\widehat{f}_{\alpha\beta}$ for the sheaf $\widehat{\mathcal{A}}$. Instead of equipping flat metrics on the U_α , one may as well start with a globally defined, curved metric on X . One can then flatten it over each U_α to obtain a free $\beta\gamma$ system. In this process the perturbative corrections disappear locally, but the transition functions change by $\widehat{f}_{\alpha\beta} \rightarrow e^{-A_\alpha} \widehat{f}_{\alpha\beta} e^{A_\beta}$ for some operators A_α . These operators carry the information on the perturbative corrections.

But then, how can we find the A_α ? Here we come back to the original problem: to do that, we need detailed knowledge of the perturbative corrections. Instead, what we can do is take a collection of locally defined free $\beta\gamma$ systems and glue them using various choices of the transition functions. In doing so, we are effectively parametrizing the theory by the way this gluing is done. This is the strategy we will adopt.

3.2 Gluing $\beta\gamma$ Systems

We are thus led to the problem of listing the possible sets of transition functions for $\widehat{\mathcal{A}}$. The first step is to classify the automorphisms of the free $\beta\gamma$ system. Automorphisms are generated by currents of dimension one. There are two types of such currents.

Let V be a holomorphic vector field on \mathbb{C}^d . Then $J_V = -V^i(\gamma)\beta_i$ is a good current. (Here and below, normal ordering is implicit in expressions containing both β_i and γ^i .) From the OPEs

$$J_V(z)\gamma^i(w) \sim \frac{V^i(w)}{z-w}, \quad J_V(z)\beta_i(w) \sim -\frac{\partial_i V^j \beta_j(w)}{z-w}, \quad (43)$$

we see that J_V generates the infinitesimal diffeomorphism $\delta\gamma = V$ and β transforms as a $(1,0)$ -form under this symmetry. We denote the corresponding conserved charge by K_V .

With a holomorphic one-form B on \mathbb{C}^d , we can also make $J_B = B_i(\gamma)\partial_z\gamma^i$. This has the OPEs

$$J_B(z)\gamma^i(w) \sim 0, \quad J_B(z)\beta_i(w) \sim -\frac{B_i(w)}{(z-w)^2} + \frac{C_{ij}\partial_z\gamma^j(w)}{z-w} \quad (44)$$

with $C = \partial B$, so generates the transformation $\delta\beta = -i_{\partial_z\gamma}C$. For any closed holomorphic two-form C , there is a holomorphic one-form B such that $C =$

∂B . Moreover, $\delta\beta = 0$ if and only if $C = 0$. The automorphisms of this type are therefore labeled by the closed holomorphic two-forms C . We denote their charges by K_C .

One can readily work out the commutators between the conserved charges. Computing the relevant OPEs, one finds

$$\begin{aligned} [K_V, K_{V'}] &= K_{[V, V']} + K_{C(V, V')}, \\ [K_V, K_C] &= K_{\mathcal{L}_V C}, \\ [K_C, K_{C'}] &= 0. \end{aligned} \tag{45}$$

Here $C(V, V') = \partial_i \partial_k V^l \partial_j \partial_l V'^k d\phi^i \wedge d\phi^j$. The last two of these relations show that the K_C generate an abelian subalgebra on which holomorphic reparametrizations act naturally.

Now, suppose that the complex manifold X is built up by gluing open patches $U_\alpha \cong \mathbb{C}^d$ using transition functions $f_{\alpha\beta}$. Transition functions for $\hat{\mathcal{A}}$ are constructed by lifting $f_{\alpha\beta}$ to automorphisms $\hat{f}_{\alpha\beta}$ of the $\beta\gamma$ system compatible with the cocycle condition. So we choose $\hat{f}_{\alpha\beta}$ such that they act on γ by $f_{\alpha\beta}$ and satisfy $\hat{f}_{\alpha\beta} \hat{f}_{\beta\gamma} \hat{f}_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$.

Naively, one might think that the cocycle condition is satisfied if one picks a cocycle $\{C_{\alpha\beta}\}$ of closed holomorphic two-forms and let $\hat{f}_{\alpha\beta}$ act on β by the pullback $f_{\alpha\beta}^*$ followed by $\exp(K_{C_{\alpha\beta}})$. The situation is actually more complicated. The commutator between two K_V s differs from the expected form by a K_C term, implying

$$\hat{f}_{\alpha\beta} \hat{f}_{\beta\gamma} \hat{f}_{\gamma\alpha} = \exp(K_{C_{\alpha\beta\gamma}}) \tag{46}$$

for some $C_{\alpha\beta\gamma}$. We need to adjust $\hat{f}_{\alpha\beta}$ so that $C_{\alpha\beta\gamma}$ disappear. The freedom at our disposal is to transform $\hat{f}_{\alpha\beta} \rightarrow \exp(K_{C'_{\alpha\beta}}) \hat{f}_{\alpha\beta}$, which shifts

$$C_{\alpha\beta\gamma} \rightarrow C_{\alpha\beta\gamma} + (\delta C')_{\alpha\beta\gamma}. \tag{47}$$

Unless $C_{\alpha\beta\gamma}$ can be canceled by an appropriate choice of $C'_{\alpha\beta}$, the gluing cannot be carried out consistently. In other words, there may be an obstruction to the existence of a sheaf of $\beta\gamma$ systems.

The obstruction has a natural physical interpretation. It is not hard to show that $C_{\alpha\beta\gamma}$ are totally antisymmetric in α, β, γ and obey $(\delta C)_{\alpha\beta\gamma\delta} = 0$, hence define a cocycle. The freedom (47) then means that the obstruction is encoded in the cohomology class

$$[C_{\alpha\beta\gamma}] \in H^2(X, \Omega_X^{2, cl}), \tag{48}$$

where $\Omega_X^{2,cl}$ is the sheaf of closed holomorphic two-forms on X . It can be shown [40] that this class is mapped to $p_1(X) \in H^4(X, \mathbb{R})$ under the Čech–Dolbeault isomorphism. Thus, the obstruction vanishes if and only if $p_1(X) = 0$. This is the condition for the perturbative cancellation of sigma model anomaly. Since the obstruction arises in lifting the diffeomorphisms used to construct the underlying manifold X , in this way we see that the sigma model anomaly causes a gravitational anomaly on the target space. The cancellation of the anomaly by adjusting $\hat{f}_{\alpha\beta}$ is a kind of the Green–Schwarz mechanism.

Given $f_{\alpha\beta}$, the choice of $\hat{f}_{\alpha\beta}$ is not unique. Suppose that we choose different transition functions, $\hat{f}'_{\alpha\beta}$. Then the two sets of transition functions are related by $\hat{f}'_{\alpha\beta} = \exp(K_{C'_{\alpha\beta}})\hat{f}_{\alpha\beta}$ for some cocycle $\{C'_{\alpha\beta}\}$. If this cocycle is exact, $C'_{\alpha\beta} = (\delta C''_{\alpha\beta})_{\alpha\beta}$, then we have $\hat{f}'_{\alpha\beta} = \exp(-K_{C''_{\alpha\beta}})\hat{f}_{\alpha\beta}\exp(K_{C''_{\beta\alpha}})$ and these define the same gluing. Hence, inequivalent choices of the transition functions are parametrized by

$$H^1(X, \Omega_X^{2,cl}). \quad (49)$$

The origin of this moduli space can be understood as follows. For any closed form \mathcal{H} of type $(3,0) \oplus (2,1)$, locally we can find a $(2,0)$ -form T such that $\mathcal{H} = dT$.⁵ Thus we have the short exact sequence

$$0 \longrightarrow \Omega_X^{2,cl} \longrightarrow \mathcal{A}_X^{2,0} \xrightarrow{d} \mathcal{Z}_X^{3,0} \oplus \mathcal{Z}_X^{2,1} \longrightarrow 0, \quad (50)$$

where $\mathcal{A}_X^{p,q}$ and $\mathcal{Z}_X^{p,q}$ are respectively the sheaves of (p,q) -forms and closed (p,q) -forms on X . Since $H^p(X, \mathcal{A}_X^{2,0}) = 0$ for $p > 0$, the long exact sequence of cohomology implies

$$H^1(X, \Omega_X^{2,cl}) \cong H^0(X, \mathcal{Z}_X^{3,0} \oplus \mathcal{Z}_X^{2,1})/dH^0(X, \mathcal{A}_X^{2,0}). \quad (51)$$

This is the space of closed forms \mathcal{H} of type $(3,0) \oplus (2,1)$ modulo those that can be written as $\mathcal{H} = dT'$ with a globally defined $(2,0)$ -form T' . It actually parametrizes the conformally invariant Q -closed term

$$\int_{\Sigma} d^2z \{Q, T_{ij} \rho_{\bar{z}}^i \partial_z \phi^j\} = \int_{\Sigma} d^2z \alpha^{\bar{k}} \mathcal{H}_{\bar{k}ij} \rho_{\bar{z}}^i \partial_z \phi^j + i \int_{\Sigma} \phi^* T, \quad (52)$$

⁵By the Poincaré lemma, locally $\mathcal{H} = d(U + V)$ for some $(2,0)$ -form U and $\bar{\partial}$ -closed $(1,1)$ -form V . By the $\bar{\partial}$ -Poincaré lemma, locally $V = \bar{\partial}W$ for some $(1,0)$ -form W . Then $T = U + V - dW$.

which depends on T only through \mathcal{H} perturbatively. We can add this term to our action. Doing so deforms the chiral algebra unless T is globally defined. Still, we can always set T to zero locally by subtracting a globally defined T' . Combined with flattening the metric, this turns the action locally into the free theory form. Therefore, the effect of this term can be treated—and is necessarily included—in the present framework.

Up to this point, we have implicitly worked in some coordinate neighborhood of Σ . The sheaf $\hat{\mathcal{A}}$ of chiral algebras in this case (when $\Sigma \cong \mathbb{C}$) is known as a sheaf of chiral differential operators [17, 40]. In order to reconstruct the chiral algebra globally on Σ , one has to glue sheaves of chiral differential operators patch by patch over Σ . This amounts to gluing free $\beta\gamma$ systems over $\Sigma \times X$. The obstruction thus takes values in

$$H^2(\Sigma \times X, \Omega_{\Sigma \times X}^{2,cl}). \quad (53)$$

A part of it depends on both Σ and X , and corresponds to the $c_1(\Sigma)c_1(X)/2$ anomaly. The moduli of the sheaf of chiral algebras are parametrized by

$$H^1(\Sigma \times X, \Omega_{\Sigma \times X}^{2,cl}). \quad (54)$$

3.3 Conformal Anomaly

Previously, we claimed that the chiral algebra lacks invariance under holomorphic reparametrizations when $c_1(X) \neq 0$, arguing that in that case perturbative corrections would annihilate the classical Q -cohomology class $[T_{zz}]$ together with another class $[\partial_z \theta]$ via the relation $[Q, T_{zz}] = \partial_z \theta$. However, we could not really exclude the possibility that θ turns out to be zero and T_{zz} remains Q -closed. Let us check that this does not happen using the tool developed in this section.

Let $\{U_\alpha\}$ be a good cover of X . On each U_α , we put a free $\beta\gamma$ system with energy-momentum tensor $T_\alpha = -\beta_{\alpha i} \partial_z \gamma_\alpha^i$. The OPE

$$J_V(z)T_\alpha(w) \sim -\frac{\partial_i V^i(w)}{(z-w)^3} - \frac{(\partial_z \partial_i V^i + V^i \beta_i)(w)}{(z-w)^2} - \frac{1}{2} \frac{\partial_z^2 (\partial_i V^i)(w)}{z-w} \quad (55)$$

shows that T_α transform as $\delta T_\alpha = -\partial_z^2 \partial_i V^i / 2$ under infinitesimal diffeomorphisms $\delta \gamma = V$. The finite form of this transformation is [40]

$$T_\beta - T_\alpha = -\frac{1}{2} \partial_z^2 \log \det \frac{\partial \gamma_\beta}{\partial \gamma_\alpha}. \quad (56)$$

Here $\partial\gamma_\beta/\partial\gamma_\alpha$ is the Jacobian matrix. We define a cocycle $\{\theta_{\alpha\beta}\}$ by

$$\theta_{\alpha\beta} = -\frac{1}{2}\partial_z \log \det \frac{\partial\gamma_\beta}{\partial\gamma_\alpha} \quad (57)$$

so that it satisfies $T_\beta - T_\alpha = \partial_z \theta_{\alpha\beta}$. Via the Čech- Q isomorphism, this equation translates to $[Q, T_{zz}] = \partial_z \theta$ for some local operator θ .

The explicit form of θ can be found as follows. Write $\theta_{\alpha\beta} = W_\beta - W_\alpha$, where W_α is the quantity $W = \partial_z \phi^i \partial_i \log \det g/2$ evaluated in the coordinate patch U_α . Since W_α transform by holomorphic transition functions, $\bar{\partial}W$ is globally defined. This gives θ as a global section of the sheaf of free $\beta\gamma$ systems, for which $Q = \bar{\partial}$. Noting $R_{i\bar{j}} = -\partial_i \bar{\partial}_{\bar{j}} \log \det g$, we find

$$\theta = -\frac{1}{2} R_{i\bar{j}} \partial_z \phi^i \alpha^{\bar{j}}. \quad (58)$$

As a globally defined local operator of the original theory, this formula gets higher order corrections since $Q = \bar{\partial}$ only to leading order. The form of θ is in accord with the previous discussion.

We introduced the perturbative Q -cohomology class $[\theta]$ through the action of Q on T_{zz} . We can also construct it as follows, mimicking the definition of the first Chern class. Let $f_{\alpha\beta}$ be transition functions of K_X . The cocycle condition $f_{\alpha\beta} f_{\beta\gamma} f_{\gamma\alpha} = 1$ implies that $(\delta \log f)_{\alpha\beta\gamma}$ are integer multiples of $2\pi i$. Thus, by applying $\delta \log / 2\pi i$ on $f_{\alpha\beta}$, we obtain an element of $H^2(X, \mathbb{Z})$. It is $c_1(X)$. To obtain an element of the chiral algebra, we apply $-\partial_z \log / 2$ instead. This gives $-\partial_z \log f_{\alpha\beta} / 2 = \theta_{\alpha\beta}$, which represents $[\theta] \in H^1(X, \hat{\mathcal{A}})$.

3.4 \mathbb{P}^1 Model

To conclude the discussion of the sheaf theory approach, let us compute the perturbative chiral algebra of the twisted model with target space $X = \mathbb{P}^1$ for the first few dimensions. We will work locally on Σ .

The target space $\mathbb{P}^1 \cong \mathbb{C} \cup \{\infty\}$ is covered by two patches, $U = \mathbb{P}^1 \setminus \{\infty\}$ with coordinate γ and $U' = \mathbb{P}^1 \setminus \{0\}$ with coordinate γ' , related to each other by

$$\gamma' = \frac{1}{\gamma}. \quad (59)$$

Classically β transforms as $\beta' = -\gamma^2 \beta$, but this formula gets corrected quantum mechanically. The quantum transformation law turns out to be

$$\beta' = -\gamma^2 \beta + 2\partial_z \gamma. \quad (60)$$

The additional term is needed to keep the $\beta\beta$ OPE regular under this transformation. Since the moduli space $H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^{2,cl}) = 0$, this is essentially the only way to glue the two free $\beta\gamma$ systems.

We first look at the zeroth Q -cohomology group \mathcal{A}^0 . The elements of \mathcal{A}^0 are represented by global sections of the sheaf $\widehat{\mathcal{A}}$ of chiral algebras on \mathbb{P}^1 .

At dimension zero, relevant local operators are holomorphic functions. Since a holomorphic function on a compact complex manifold must be constant, the dimension zero subspace of \mathcal{A}^0 is one-dimensional and generated by the cohomology class [1], represented by the identity operator 1.

At dimension one, the possible local operators are those of the form $B(\gamma)\partial_z\gamma$ and $V(\gamma)\beta$, where B is a holomorphic one-form and V is a holomorphic vector field. There are no global holomorphic one-forms on \mathbb{P}^1 . For holomorphic vector fields, we have three independent ones, ∂ , $-\gamma\partial$, and $-\gamma^2\partial$. Hence, at the classical level we have three cohomology classes, represented by β , $-\gamma\beta$, and $-\gamma^2\beta$. These survive to the perturbative chiral algebra. Their quantum counterparts are

$$\begin{aligned} J_- &= \beta = -\gamma'^2\beta' + 2\partial_z\gamma', \\ J_3 &= -\gamma\beta = \gamma'\beta', \\ J_+ &= -\gamma^2\beta + 2\partial_z\gamma = \beta', \end{aligned} \tag{61}$$

generating the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ at the critical level -2 :

$$\begin{aligned} J_3(z)J_3(w) &\sim -\frac{1}{(z-w)^2}, \\ J_3(z)J_{\pm}(w) &\sim \pm\frac{J_{\pm}(w)}{(z-w)^2}, \\ J_+(z)J_-(w) &\sim -\frac{2}{(z-w)^2} + \frac{2J_3(w)}{z-w}. \end{aligned} \tag{62}$$

The existence of these currents in the perturbative chiral algebra is a reflection of the fact that \mathbb{P}^1 admits an SL_2 -action.

We now turn to the first Q -cohomology group \mathcal{A}^1 . The elements of \mathcal{A}^1 are represented by sections of $\widehat{\mathcal{A}}(U \cap U')$ that cannot be written as the difference of a section of $\widehat{\mathcal{A}}(U)$ and a section of $\widehat{\mathcal{A}}(U')$. Extended over the whole target space \mathbb{P}^1 , such sections necessarily have poles at both 0 and ∞ .

Meromorphic functions with poles at 0 and ∞ can always be split into a part regular at 0 and a part regular at ∞ . Thus the dimension zero subspace of \mathcal{A}^1 is zero.

At dimension one, we can try operators of the form β/γ^n which have a pole at 0. However, they are all regular at ∞ . The other possibilities are $\partial_z\gamma/\gamma^n$. Requiring they have a pole at ∞ , we find that only $\partial_z\gamma/\gamma$ can represent a nontrivial cohomology class. Indeed, it does. This cohomology class is $[\theta]$ since $\partial_z\gamma/\gamma = -\partial_z \log(\partial\gamma'/\partial\gamma)/2$.

At dimension two, sections with poles at both 0 and ∞ are linear combinations of $\partial_z^2\gamma/\gamma$, $\partial_z^2\gamma/\gamma^2$, $(\partial_z\gamma)^2/\gamma$, $(\partial_z\gamma)^2/\gamma^2$, $(\partial_z\gamma)^2/\gamma^3$, and $\beta\partial_z\gamma/\gamma$. Among these, the combinations $\partial_z^2\gamma/\gamma^2 - 2(\partial_z\gamma)^2/\gamma^3$ and $\beta\partial_z\gamma/\gamma + (\partial_z\gamma)^2/\gamma^3$ are regular at ∞ , so vanish in the cohomology. (In verifying this assertion, one should keep in mind that the latter operator is normal ordered.) Moreover, $\partial_z(\partial_z\gamma/\gamma)$ also vanishes due to the perturbative relation $[Q, T_{zz}] = \partial_z\theta$. Thus the dimension two subspace of \mathcal{A}^1 is at most three-dimensional. From the cohomology classes we already have, we can construct three: $[J_-\theta]$, $[J_3\theta]$, and $[J_+\theta]$.

Let us summarize. The dimension zero subspace of \mathcal{A}^0 is generated by $[1]$ and the dimension one subspace of \mathcal{A}^1 is generated by $[\theta]$, whereas the dimension one subspace of \mathcal{A}^0 is generated by $[J_-]$, $[J_3]$, $[J_+]$ and the dimension two subspace of \mathcal{A}^1 is generated by $[J_-\theta]$, $[J_3\theta]$, $[J_+\theta]$. Therefore, for the first two nontrivial dimensions, we find an isomorphism $\mathcal{A}^0 \cong \mathcal{A}^1$ given by the map

$$[\mathcal{O}] \mapsto [\mathcal{O}\theta]. \quad (63)$$

It has been shown [17] that this isomorphism persists in higher dimensions. This is as though $[1]$ and $[\theta]$ are vacua of conformal field theory—both of them are annihilated by ∂_z —and the elements of \mathcal{A}^0 are creation operators acting on these classes to generate the rest of the Q -cohomology classes. It is remarkable that such a structure emerges despite the lack of conformal invariance. In order to understand where this structure comes from, we must go beyond perturbation theory.

4 Nonperturbative Vanishing of Chiral Algebras

In the previous sections we have seen that quantum corrections deform the classical chiral algebra, but perturbatively the deformation can be understood within the framework of a cohomology theory on the target space. This is because in perturbation theory, one considers fluctuations localized around

constant maps.

Instantons are not quite like constant maps, but have a finite size in the target space. Their presence may therefore lead to deformations of different kinds. In this section, we will see a particularly striking example: instantons annihilate all of the perturbative Q -cohomology classes, making the chiral algebra trivial nonperturbatively. The existence of such a phenomenon was first predicted by Witten [16], and subsequently confirmed by Tan and the author [23]; see Arakawa and Malikov [41] for a mathematical interpretation of Witten’s prediction. Here we generalize the results of [23, 24, 25] and establish a vanishing “theorem” for chiral algebras. We then explain how the vanishing of the chiral algebra implies the spontaneous breaking of supersymmetry and the absence of harmonic spinors on the loop space of the target space.

4.1 \mathbb{P}^1 Model, with Instantons

The simplest example of a vanishing chiral algebra is provided by the \mathbb{P}^1 model which we studied in Section 3.4. As we saw there, the perturbative chiral algebra of the \mathbb{P}^1 model has the structure of a Fock space: $[1]$ and $[\theta]$ play the role of “ground states,” on which infinite towers of bosonic and fermionic “excited states” are constructed by acting with “creation operators,” namely bosonic Q -cohomology classes. In view of this suggestive structure, one may expect that instantons “tunnel” between the “ground states” and “lift” them out of the chiral algebra. This is indeed the case.

We now show that the perturbative Q -cohomology classes $[1]$ and $[\theta]$ are annihilated together by instantons via the relation

$$\{Q, \theta\} \propto 1. \tag{64}$$

This relation says that the equation $1 = 0$ holds in the Q -cohomology. Therefore, the chiral algebra of the \mathbb{P}^1 model vanishes nonperturbatively. A half of the perturbative Q -cohomology classes are annihilated because their representatives become Q -exact. For classes $[\mathcal{O}]$ in this category, we have $\{Q, \mathcal{O}\theta\} \propto \mathcal{O}$. Thus $[\mathcal{O}]$ are annihilated together with $[\mathcal{O}\theta]$, and the latter should constitute the other half of the perturbative chiral algebra. This explains the observed Fock space structure.

Since $c_1(\mathbb{P}^1) = 2$, instantons of degree k (which wrap the target space k times) violate R-charge by $2k$ and dimension by k . Then the instanton

corrections to the action of Q on θ take the form

$$\{Q, \theta\} = \sum_{k=1}^{\infty} \Lambda^{2k} e^{-kt_0} \mathcal{O}_k, \quad (65)$$

with \mathcal{O}_k being local operators of R-charge $2 - 2k$ and dimension $(1 - k, 0)$. Here t_0 is the topological invariant S_B evaluated for instantons of degree one at the dynamical scale Λ . There are no local operators of negative dimension, hence $\mathcal{O}_k = 0$ for $k > 1$. The remaining operator, \mathcal{O}_1 , is perturbatively Q -closed. From the fact that for R-charge zero and dimension zero, $[1]$ is the only perturbative Q -cohomology class and there are no perturbatively Q -exact local operators, it follows $\mathcal{O}_1 \propto 1$. So if we can show $\{Q, \theta\} \neq 0$, we establish the relation $\{Q, \theta\} \propto 1$.

To see whether $\{Q, \theta\}$ is zero or not, we put it on a disk (embedded in the worldsheet) and evaluate the path integral for suitable boundary conditions. For our purpose, we can compactify the disk to the sphere and consider those boundary conditions that can be represented by vertex operators at ∞ . We will therefore compute the correlation function

$$\left\langle \mathcal{O}(\infty) \oint d\bar{z} G(\bar{z}) \theta(0) \right\rangle \quad (66)$$

on $\Sigma = \mathbb{P}^1$ for some local operators \mathcal{O} . We anticipate that $\{Q, \theta\}$, expressed here as the contour integral of the supercurrent G around θ , will be replaced by 1 in the final result. To obtain a nonvanishing answer, then, we should take \mathcal{O} to be local operators of R-charge zero and dimension zero, that is, functions on X .

The contributions to this correlation function should come from instantons of degree one. These are biholomorphic maps from $\Sigma = \mathbb{P}^1$ to $X = \mathbb{P}^1$, whose moduli space $\mathcal{M}_1 \cong \text{PGL}_2(\mathbb{C})$. At $\phi_0 \in \mathcal{M}_1$, the number of $\bar{\psi}_+$ and ψ_+ zero modes are respectively given by the zeroth and first Hodge numbers of the bundle $K_\Sigma^{1/2} \otimes \phi_0^* T_X \cong \mathcal{O}_{\mathbb{P}^1}(1)$. Notice that we have untwisted the theory before compactifying, because the twisted \mathbb{P}^1 model is anomalous on \mathbb{P}^1 . Since $h^0(\mathcal{O}_{\mathbb{P}^1}(1)) = 2$ and $h^1(\mathcal{O}_{\mathbb{P}^1}(1)) = 0$, there are two $\bar{\psi}_+$ zero modes and no ψ_+ zero modes. These zero modes can be absorbed by the $\bar{\psi}_+$ fields in G and θ without bringing down interaction terms. Then, up to the ratio of the bosonic and fermionic determinants, the correlation function is computed to leading order by dropping quantum fluctuations and integrating over the

fermion zero modes as well as the instantons:

$$e^{-S_B(\phi_0)} \int d\mathcal{M}_1 dc_0^1 dc_0^2 \mathcal{O}(\infty) \oint d\bar{z} G(\bar{z}) \theta(0) \Big|_{\phi=\phi_0}. \quad (67)$$

Here $d\mathcal{M}_1$ is the measure on \mathcal{M}_1 , and c_0^1, c_0^2 are the zero mode coefficients of the mode expansion of $\bar{\psi}_+$. We can ignore subleading contributions.

Of course, we must evaluate the contour integral before dropping quantum fluctuations. (Without short distance singularities this would vanish!) It turns out that this seemingly straightforward task is actually very tricky. We are looking for an antiholomorphic single pole $1/\bar{z}$ in the OPE

$$G(\bar{z})\theta(0) = (g_{\phi\bar{\phi}}\partial_{\bar{z}}\phi\bar{\psi}_+)(\bar{z})\left(-\frac{1}{2}R_{\phi\bar{\phi}}\partial_z\phi\bar{\psi}_+\right)(0). \quad (68)$$

It may appear that one can obtain such a pole by contracting $\partial_{\bar{z}}\phi$ with $R_{\phi\bar{\phi}}$. However, this does not work because the residue is just the classical action of Q on θ and vanishes. We must find additional antiholomorphic poles that emerge nonperturbatively.

At this point, we recall that the fermionic fields take values in the pullback of the tangent bundle of X by the bosonic field. Thus, the eigenmodes in which they are expanded depend on the bosonic field, which is itself subject to quantum fluctuations. As a result, the fermion modes—even the zero modes—can produce short distance singularities when the bosonic field is present at the same location.

We can try to extract this bosonic dependence of the fermionic fields as follows. Consider a tubular neighborhood \mathcal{N}_1 of \mathcal{M}_1 , diffeomorphic to the normal bundle of \mathcal{M}_1 in the space of maps from Σ to X . Let $\{x^\alpha\}$ be local coordinates on \mathcal{M}_1 and parametrize the normal directions by coordinates $\{y^\beta\}$ such that $y^\beta = 0$ on \mathcal{M}_1 . For $\phi(z, \bar{z}; x, y) \in \mathcal{N}_1$, we denote its projection to \mathcal{M}_1 by $\phi_0(z; x)$. An instanton $\phi_0 \in \mathcal{M}_1$ maps the points of $\Sigma = \mathbb{P}^1$ to the points of $X = \mathbb{P}^1$ in a one-to-one manner, so we can invert $\phi_0(z; x)$ to obtain $z(\phi_0; x)$ and write $\phi(z, \bar{z}; x, y) = \phi(\phi_0(z; x), \bar{\phi}_0(\bar{z}; x); x, y)$. Computing $[iQ, \bar{\phi}]$ with this last expression, we find

$$\bar{\psi}_+ = \frac{\partial \bar{\phi}}{\partial \bar{\phi}_0} [iQ, \bar{\phi}_0] + \cdots = \frac{\partial_{\bar{z}} \bar{\phi}}{\partial_{\bar{z}} \bar{\phi}_0} \bar{\psi}_{+0}(\phi_0) + \cdots, \quad (69)$$

where $\bar{\psi}_{+0}(\phi_0) = [iQ, \bar{\phi}_0]$ is the zero mode part of $\bar{\psi}_+$ evaluated at ϕ_0 . So we have extracted, partially, the dependence of $\bar{\psi}_+$ on the bosonic fluctuations.

In fact, the leading term of the formula (69) is all we need. The reason is that the fermion nonzero modes will be discarded in our computation and, up to the nonzero modes and the equation of motion for the bosonic field, the leading term represents the zero mode part of $\bar{\psi}_+$. Indeed, it correctly reduces to $\bar{\psi}_{+0}(\phi_0)$ at $\phi = \phi_0$ and the equation of motion implies

$$D_z \left(\frac{\partial_{\bar{z}} \bar{\phi}}{\partial_{\bar{z}} \bar{\phi}_0} \bar{\psi}_{+0}(\phi_0) \right) = R^{\bar{\phi}}_{\bar{\phi} \phi \bar{\phi}} \frac{\partial_z \bar{\phi}}{\partial_{\bar{z}} \bar{\phi}_0} \psi_+ \bar{\psi}_+ \bar{\psi}_{+0}(\phi_0), \quad (70)$$

which vanishes if the fermion nonzero modes are dropped since there are no ψ_+ zero modes.

As desired, $\partial_{\bar{z}} \phi$ from G can now be contracted with $\partial_{\bar{z}} \bar{\phi}$ in the $\bar{\psi}_+$ field from θ to produce an antiholomorphic double pole. This gives

$$\oint d\bar{z} G(\bar{z}) \theta(0) = \left(\frac{i}{2} R_{\phi \bar{\phi}} \frac{\partial_z \phi}{\partial_{\bar{z}} \bar{\phi}_0} \partial_{\bar{z}} \bar{\psi}_+ \bar{\psi}_{+0}(\phi_0) \right) (0). \quad (71)$$

The result may look strange, but will become more natural after the $\bar{\psi}_+$ zero modes are integrated out.

To proceed, we need to specify the path integral measure. On instantons, the action of Q is realized as the superconformal transformation $\bar{z} \mapsto \bar{z} + \bar{\epsilon}_-(c_0^1 + c_0^2 \bar{z})$. Thus the zero mode part of $\bar{\psi}_+$ can be expanded as

$$\bar{\psi}_{+0}(\phi_0) = c_0^1 \partial_{\bar{z}} \bar{\phi}_0 + c_0^2 \bar{z} \partial_{\bar{z}} \bar{\phi}_0. \quad (72)$$

If we choose $d\mathcal{M}_1$ to be conformally invariant, it is Q -invariant up to terms involving dc_0^1 or dc_0^2 . Then the product $d\mathcal{M}_1 dc_0^1 dc_0^2$ is a Q -invariant measure since $dc_0^1 dc_0^2$ is Q -invariant by itself. There is a unique conformally invariant measure on \mathcal{M}_1 up to a factor. In terms of the points $X_0, X_1, X_\infty \in X$ to which $0, 1, \infty \in \Sigma$ are mapped by instantons, it is given by

$$d\mathcal{M}_1 = \frac{d^2 X_0 d^2 X_1 d^2 X_\infty}{|X_0 - X_1|^2 |X_1 - X_\infty|^2 |X_\infty - X_0|^2}. \quad (73)$$

The parametrization of \mathcal{M}_1 by X_0, X_1, X_∞ provides a compactification of \mathcal{M}_1 to $(\mathbb{P}^1)^3$, but $d\mathcal{M}_1$ is singular on the compactified moduli space.

Let us return to the integral (67). After the integration over the $\bar{\psi}_+$ zero modes, the contour integral

$$\int dc_0^1 dc_0^2 \oint d\bar{z} G(\bar{z}) \theta(0) \Big|_{\phi=\phi_0} = \left(\frac{i}{2} R_{\phi \bar{\phi}} \partial_z \phi_0 \partial_{\bar{z}} \bar{\phi}_0 \right) (0). \quad (74)$$

This is the pullback of the Ricci form by instantons. Using the formula

$$\partial_z \phi_0(0) = \frac{(X_\infty - X_0)(X_0 - X_1)}{X_1 - X_\infty}, \quad (75)$$

what is left can be written as

$$\frac{i}{2} e^{-S_B(\phi_0)} \int_{\mathbb{P}^1} d^2 X_0 R_{\phi\bar{\phi}}(X_0, \bar{X}_0) \int_{(\mathbb{P}^1)^2} d^2 X_1 d^2 X_\infty \frac{\mathcal{O}(X_\infty, \bar{X}_\infty)}{|X_1 - X_\infty|^4}. \quad (76)$$

The X_0 -integral is the evaluation of $2\pi c_1(X)$ on $[X]$, which gives 4π . The X_1 -integral diverges, reflecting the noncompactness of \mathcal{M}_1 . One way to regularize it is to impose a lower bound $l > 0$ on the distance between X_1 and X_∞ measured with the target space metric. For $l \ll 1$, this restricts the domain of X_1 to the region where $g_{\phi\bar{\phi}}(X_\infty, \bar{X}_\infty)|X_1 - X_\infty|^2 \geq l^2$. The regularized integral is

$$\int_{\mathbb{P}^1} \frac{d^2 X_1}{|X_1 - X_\infty|^4} = \frac{\pi}{l^2} g_{\phi\bar{\phi}}(X_\infty, \bar{X}_\infty). \quad (77)$$

From this and the renormalization group equation (23) for the B field, we find that the integral (76) contains a factor $(\Lambda/l\mu)^2 e^{-t_0}$. We can take a limit such that $\mu \rightarrow \infty$ and $l \rightarrow 0$ keeping $l\mu$ finite. The end result is

$$\Lambda^2 e^{-t_0} \int_{\mathbb{P}^1} g_{\phi\bar{\phi}} d^2 X_\infty \mathcal{O}, \quad (78)$$

up to an overall numerical factor.

We see that the resulting X_∞ -integral is performed over $\mathcal{M}_0 \cong X$ with respect to a natural volume form. Therefore, the one-instanton computation of the correlation function (66) has reduced to the zero-instanton computation of the expected one-point function:

$$\langle \mathcal{O}(\infty) \{Q, \theta(0)\} \rangle \propto \Lambda^2 e^{-t_0} \langle \mathcal{O}(\infty) \rangle. \quad (79)$$

This shows $\{Q, \theta\} \propto 1$.

4.2 Vanishing “Theorem”

The vanishing of the chiral algebra is not special to the \mathbb{P}^1 model. It seems to be a feature shared by many $(0,2)$ models. In fact, we have the following vanishing “theorem.” Let X be a compact spin Kähler manifold with

$p_1(X)/2 = 0$ and $c_1(X) > 0$. Suppose that there is an embedding $\mathbb{P}^1 \subset X$ with trivial normal bundle. Then, the chiral algebra of the $(0, 2)$ model with target space X vanishes nonperturbatively in the absence of left-moving fermions.

As before, we will establish the vanishing by demonstrating that instantons annihilate the perturbative Q -cohomology classes $[1]$ and $[\theta]$ together. Since $c_1(X) > 0$, every instanton violates R-charge by $2k$ and dimension by k for some integer $k > 0$. Then $[\theta]$ can only be annihilated by instantons with $k = 1$, in which case it is paired with a class of R-charge zero and dimension zero. Such a class must be proportional to $[1]$ on the compact complex manifold X . Thus, it suffices to show $\{Q, \theta\} \neq 0$.

Consider the correlation function (66) again. The leading contributions to this function come from instantons that give precisely two $\bar{\psi}_+$ zero modes and no ψ_+ zero modes. These are embeddings $\mathbb{P}^1 \hookrightarrow X$ whose normal bundle is trivial. (The normal bundle $N_{C/X}$ of a curve $C \subset X$ is a holomorphic vector bundle over C defined by the short exact sequence $0 \rightarrow T_C \rightarrow T_X|_C \rightarrow N_{C/X} \rightarrow 0$.) To see this, note that given instantons wrapping a $\mathbb{P}^1 \subset X$ once, there are already the right amount of zero modes from the tangent direction. So there should be no additional zero modes from the normal directions, which is the case if and only if the normal bundle of the \mathbb{P}^1 is trivial.

But if the normal bundle is trivial and the fermion zero modes come only from the tangent direction, the contribution from each such \mathbb{P}^1 can be computed essentially in the same way as in the \mathbb{P}^1 model; the result is a function on X supported on the \mathbb{P}^1 . By assumption, there is at least one \mathbb{P}^1 that makes a contribution, hence the correlation function is nonzero.⁶ It follows that $\{Q, \theta\} \neq 0$ and the chiral algebra vanishes.

One may wonder how the existence of a single \mathbb{P}^1 with trivial normal bundle, whose contribution to $\{Q, \theta\}$ is confined on the \mathbb{P}^1 itself, can possibly tell something about the behavior of $\{Q, \theta\}$ in the other region of X . This point can be understood if we consider deformations of the \mathbb{P}^1 . Infinitesimal deformations are given by holomorphic sections of the normal bundle. Since the normal bundle is trivial, the \mathbb{P}^1 in question can be moved in every direction in the target space. Then the family of \mathbb{P}^1 s generated by deformations

⁶In case the contributions from all the instantons add up to zero, we can insert delta functions supported on a particular \mathbb{P}^1 at various points on Σ so that the target space effectively becomes that \mathbb{P}^1 . The correlation function is still nonzero.

sweep out the whole target space, and their contributions can add up to a nonzero constant.

4.3 Flag Manifold Model

An example in which the above deformation argument is beautifully demonstrated is the flag manifold G/B of a complex simple Lie group G with Borel subgroup B [24, 25]. For $X = G/B$, we have $p_1(X)/2 = 0$ and $c_1(X) = 2(x_1 + \cdots + x_r)$ with $r = \text{rank } G$. Thus the G/B model is well defined and has the R-symmetry broken to \mathbb{Z}_2 nonperturbatively. The simplest case is when $G = \text{SL}_2$, for which $G/B \cong \mathbb{P}^1$.

What makes the G/B model interesting is that its perturbative chiral algebra contains currents generating the affine Lie algebra $\widehat{\mathfrak{g}}$ of critical level [17, 26, 27]. The critical level makes an appearance here because the Sugawara construction must fail; otherwise, it would contradict the fact that the chiral algebra lacks the energy-momentum tensor when $c_1(X) \neq 0$. Nonperturbatively, these currents disappear from the chiral algebra along with everything else.

Pick a $\mathbb{P}^1 \subset X$ and call it C . Choose linearly independent normal vectors $V_1, \dots, V_d \in N_{C/X}|_{g_0}$ at a point $g_0 \in C$. Under $g_0 \mapsto gg_0 \in C$, these are mapped to $g_*V_1, \dots, g_*V_d \in N_{C/X}|_{gg_0}$ which are again linearly independent. Varying g , we obtain a global frame of $N_{C/X}$. Therefore $N_{C/X}$ is trivial, and the chiral algebra vanishes. In this example, the G -action generates a family of \mathbb{P}^1 s with trivial normal bundle that covers the target space.

4.4 Supersymmetry Breaking and Geometry of Loop Spaces

The vanishing of the chiral algebra of a $(0, 2)$ model has important implications for the dynamics of the theory and the geometry of the loop space $\mathcal{L}X$ of the target space: supersymmetry is spontaneously broken and there are no harmonic spinors on $\mathcal{L}X$. These conclusions are obtained by studying the Q -cohomology of states rather than operators.

Since Q has R-charge one and satisfies $Q^2 = 0$, one can consider the Q -cohomology graded by the R-charge in the Hilbert space of states. This is naturally a module over the chiral algebra: on elements $[[\Psi]]$ of the Q -

cohomology of states, $[\mathcal{O}] \in \mathcal{A}$ acts by

$$[\mathcal{O}] \cdot [|\Psi\rangle] = [\mathcal{O}|\Psi\rangle]. \quad (80)$$

As a graded vector space, it is isomorphic to the space of supersymmetric states, the kernel of $\{Q, Q^\dagger\} \propto H - P$ which is generally infinite-dimensional since P is not bounded. If $c_1(X) = 0$, it is further isomorphic to the Q -cohomology of local operators by the state-operator correspondence.

To unravel the geometric meaning of the Q -cohomology of states, take Σ to be a cylinder $S^1 \times \mathbb{R}$ with coordinates (σ, τ) and regard τ as time. The theory may now be viewed as supersymmetric quantum mechanics on $\mathcal{L}X$, so let us canonically quantize it and see what we get. The fermionic fields are quantized (for $z = \sigma + i\tau$ and with respect to a local orthonormal frame) to obey

$$\{\psi_+^a(\sigma, \tau), \bar{\psi}_+^{\bar{b}}(\sigma', \tau)\} = \delta^{ab} \delta(\sigma - \sigma'). \quad (81)$$

This is a loop space version of the Clifford algebra, with the continuous index σ parametrizing the direction along the loop. States are thus spinors on $\mathcal{L}X$. The supercharge is quantized as

$$Q = -i \int d\sigma \bar{\psi}_+^{\bar{i}} \left(\frac{D}{\delta\phi^{\bar{i}}} - B_{ij} \partial_\sigma \phi^j - B_{\bar{i}\bar{j}} \partial_\sigma \phi^{\bar{j}} \right), \quad (82)$$

where $D/\delta\phi$ is the covariant functional derivative on $\mathcal{L}X$. From this expression, we see that Q is almost a half of the Dirac operator on $\mathcal{L}X$. But not quite, since it has extra pieces coupled to ∂_σ .

We can eliminate these extra pieces without changing the Q -cohomology. To do this, we define a functional $\mathcal{A}_B: \mathcal{L}X \rightarrow \mathbb{C}$ as follows.⁷ First, we pick a base loop in each connected component of $\mathcal{L}X$. Then, given $\phi \in \mathcal{L}X$, we choose a homotopy $\hat{\phi}: [0, 1] \times S^1 \rightarrow X$ from the base loop of the relevant component to ϕ . Finally, we set

$$\mathcal{A}_B(\phi) = \int_{[0,1] \times S^1} \hat{\phi}^* B. \quad (83)$$

Under a variation of the end point $\phi \rightarrow \phi + \delta\phi$, this functional changes by

$$\delta\mathcal{A}_B = \int d\sigma \left(\delta\phi^i (B_{ij} \partial_\sigma \phi^j + B_{i\bar{j}} \partial_\sigma \phi^{\bar{j}}) - \delta\phi^{\bar{i}} (B_{\bar{i}j} \partial_\sigma \phi^j + B_{\bar{i}\bar{j}} \partial_\sigma \phi^{\bar{j}}) \right). \quad (84)$$

⁷This functional is not single valued if the cohomology class $[B]$ does not vanish on some two-cycles. However, one can always go to a covering space of $\mathcal{L}X$ in which it becomes single valued, and define the theory in this space. See [42] for more discussion.

Writing Q_0 for a half of the Dirac operator obtained by dropping the extra pieces from Q , we have

$$Q = e^{-A_B} Q_0 e^{A_B} . \quad (85)$$

Therefore, the Q -cohomology of states is isomorphic to the Q_0 -cohomology, which is the cohomology of the spinor bundle over $\mathcal{L}X$.

Now suppose that the chiral algebra is trivial: $[1] = 0$. Then

$$[|\Psi\rangle] = [1] \cdot [|\Psi\rangle] = 0 \quad (86)$$

for any Q -cohomology classes $[|\Psi\rangle]$, so the Q -cohomology of states is also trivial. Since supersymmetric states are in one-to-one correspondence with the Q -cohomology classes, there are none. In other words, supersymmetry is spontaneously broken. Furthermore, we just saw that the Q -cohomology is nothing but the cohomology of the spinor bundle over $\mathcal{L}X$. Hence, there are no harmonic spinors on $\mathcal{L}X$ either.

References

- [1] W. Lerche, C. Vafa, and N. P. Warner, *Chiral rings in $N = 2$ superconformal theories*, Nucl. Phys. **B 324** (1989) 427.
- [2] E. Witten, *Topological sigma models*, Comm. Math. Phys. **118** (1988) 411.
- [3] E. Witten, *Mirror manifolds and topological field theory*, in *Essays on mirror manifolds*, pp. 120–158. Int. Press, Hong Kong, 1992. [hep-th/9112056](#).
- [4] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985) 307.
- [5] A. Floer, *A relative Morse index for the symplectic action*, Comm. Pure Appl. Math. **41** (1988) 393.
- [6] A. Floer, *The unregularized gradient flow of the symplectic action*, Comm. Pure Appl. Math. **41** (1988) 775.
- [7] A. Floer, *Morse theory for Lagrangian intersections*, J. Diff. Geom. **28** (1988) 513.

- [8] A. Floer, *Symplectic fixed points and holomorphic spheres*, Comm. Math. Phys. **120** (1989) 575.
- [9] A. Floer, *Witten's complex and infinite-dimensional Morse theory*, J. Diff. Geom. **30** (1989) 207.
- [10] L. J. Dixon, *Some world-sheet properties of superstring compactifications, on orbifolds and otherwise*, in *Superstrings, unified theories and cosmology*, pp. 67–126. World Scientific, Singapore, 1988.
- [11] K. Hori and C. Vafa, *Mirror symmetry*, [hep-th/0002222](#).
- [12] E. Witten, *Elliptic genera and quantum field theory*, Comm. Math. Phys. **109** (1987) 525.
- [13] E. Witten, *Index of Dirac operators*, in *Quantum fields and strings: a course for mathematicians, vol. 1*, pp. 475–511. Amer. Math. Soc., Providence, RI, 1999.
- [14] C. Vafa, *Superstring vacua*, in *High Energy Physics and Cosmology*, pp. 145–177. World Scientific, Singapore, 1989.
- [15] E. Witten, *On the Landau–Ginzburg description of $N=2$ minimal models*, Int. J. Mod. Phys. **A 9** (1994) 4783, [[hep-th/9304026](#)].
- [16] E. Witten, *Two-dimensional models with $(0,2)$ supersymmetry: perturbative aspects*, Adv. Theor. Math. Phys. **11** (2007) 1.
- [17] F. Malikov, V. Schechtman, and A. Vaintrob, *Chiral de Rham complex*, Comm. Math. Phys. **204** (1999) 439, [[math/9803041](#)].
- [18] A. Kapustin, *Chiral de Rham complex and the half-twisted sigma-model*, [hep-th/0504074](#).
- [19] M.-C. Tan, *Two-dimensional twisted sigma models and the theory of chiral differential operators*, Adv. Theor. Math. Phys. **10** (2006) 759, [[hep-th/0604179](#)].
- [20] M.-C. Tan, *The half-twisted orbifold sigma model and the chiral de Rham complex*, Adv. Theor. Math. Phys. **12** (2008) 547, [[hep-th/0607199](#)].

- [21] M.-C. Tan, *Equivariant cohomology of the chiral de Rham complex and the half-twisted gauged sigma model*, Adv. Theor. Math. Phys. **13** (2009) 3, [[hep-th/0612164](#)].
- [22] M.-C. Tan, *Two-dimensional twisted sigma models, the mirror chiral de Rham complex, and twisted generalised mirror symmetry*, JHEP **07** (2007) 13, [[arXiv:0705.0790](#)].
- [23] M.-C. Tan and J. Yagi, *Chiral algebras of $(0, 2)$ sigma models: beyond perturbation theory*, [arXiv:0801.4782](#).
- [24] M.-C. Tan and J. Yagi, *Chiral algebras of $(0, 2)$ sigma models: beyond perturbation theory – II*, [arXiv:0805.1410](#).
- [25] M.-C. Tan and J. Yagi, *Chiral algebras of $(0, 2)$ models: beyond perturbation theory*, Lett. Math. Phys. **84** (2008) 257.
- [26] B. L. Feigin and E. V. Frenkel, *Affine Kac–Moody algebras and semi-infinite flag manifolds*, Comm. Math. Phys. **128** (1990) 161.
- [27] E. Frenkel, *Lectures on the Langlands program and conformal field theory*, in *Frontiers in number theory, physics, and geometry. II*, pp. 387–533. Springer, Berlin, 2007. [hep-th/0512172](#).
- [28] S. Stolz, *A conjecture concerning positive Ricci curvature and the Witten genus*, Math. Ann. **304** (1996) 785.
- [29] G. W. Moore and P. C. Nelson, *Anomalies in nonlinear sigma models*, Phys. Rev. Lett. **53** (1984) 1519.
- [30] G. W. Moore and P. C. Nelson, *The Aetiology of sigma model anomalies*, Comm. Math. Phys. **100** (1985) 83.
- [31] E. Witten, *Global anomalies in string theory*, in *Symposium on anomalies, geometry, topology*, pp. 61–99. World Scientific, Singapore, 1985.
- [32] M. F. Atiyah, *Circular symmetry and stationary-phase approximation*, Astérisque **131** (1985) 43.
- [33] T. P. Killingback, *World sheet anomalies and loop geometry*, Nucl. Phys. **B 288** (1987) 578.

- [34] D. Friedan, *Nonlinear models in $2 + \varepsilon$ dimensions*, Phys. Rev. Lett. **45** (1980) 1057.
- [35] D. H. Friedan, *Nonlinear models in $2 + \varepsilon$ dimensions*. Ph.D. thesis, University of California at Berkeley, August 1980. Published as Lawrence Berkeley Laboratory Publication LBL-11517 and Ann. Phys. **163** (1985) 318.
- [36] L. Alvarez-Gaume, D. Z. Freedman, and S. Mukhi, *The background field method and the ultraviolet structure of the supersymmetric nonlinear sigma model*, Ann. Phys. **134** (1981) 85.
- [37] C. G. Callan, Jr., E. J. Martinec, M. J. Perry, and D. Friedan, *Strings in background fields*, Nucl. Phys. **B 262** (1985) 593.
- [38] E. Witten, *Phases of $N = 2$ theories in two dimensions*, Nucl. Phys. **B 403** (1993) 159, [[hep-th/9301042](#)].
- [39] A. Adams, J. Distler and M. Ernebjerg, *Topological heterotic rings*, Adv. Theor. Math. Phys. **10**, (2006) 657, [[hep-th/0506263](#)].
- [40] V. Gorbounov, F. Malikov, and V. Schechtman, *Gerbres of chiral differential operators*, Math. Res. Lett. **7** (2000) 55.
- [41] T. Arakawa, F. Malikov, *A Vertex algebra attached to the flag manifold and Lie algebra cohomology*, AIP Conf. Proc. **1243** (2010) 151. [[arXiv:0911.0922](#)].
- [42] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow, *Mirror symmetry*, vol. 1 of Clay Mathematics Monographs. American Mathematical Society, Providence, RI, 2003. With a preface by Vafa.